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Effect of polydispersity in conductivity of unidirectional cylinders

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ABSTRACT

Purpose: Purpose of this paper is to study the effective thermal conductivity of infinite unidirectional circular cylinders arbitrary distributed in a uniform host.

Design/methodology/approach: Applying the method of complex potentials we present the heat flux as a complex analytic function in the section of the fibre composite perpendicular to the direction of the cylinders.

Findings: The flux is written in the form of a series. Special attention is paid to the question of polydispersity, when many cylinders of different radii are distributed in the periodicity cell.

Research limitations/implications: The proposed exact formulas hold for fiber composites.

Practical implications: The deduced formulas are useful in prediction and estimation of the effective thermal conductivity of fiber composites.

Originality/value: The obtained formulas for the effective conductivity are new and were not known before.

Keywords: Effective conductivity; Fibre composite; Polydispersity; Complex potential

METHODOLOGY OF RESEARCH, ANALYSIS AND MODELLING

1. Introduction

The present paper is addressed to the study of the unidirectional infinite circular cylinders of conductivity λ embedded in a host matrix of the normalised unit conductivity.

For definiteness, the case $\lambda > 1$ is considered, i.e., the conductivity of inclusions is greater than the conductivity of matrix. It is assumed that the section of any cylinder perpendicular to its axis is a disk. Then we arrive at the 2D problem which consists in the determination of the potential satisfying Laplace's equation with appropriate boundary or conjugation conditions on the boundaries of the disks.

Having used a method of functional equations Mityushev et al. [1-10] studied 2D problems for non-overlapping disks. A general theory of the corresponding boundary value problems were described in [4]. It was extended to periodic plane problem but for infinite number of inclusions.

The problem with one circular inclusion per periodicity cell were solved exactly in [9] in terms of the lattice sums investigated in [7,10]. The locations of disks, their number and radii in [1-3,10] were arbitrary. The local temperature fields and the effective conductivity tensor were obtained in the form of power series on Bergman's contrast parameter [11,12]. The effective conductivity for a composite medium consisting of a dense square array of identical, perfectly conducting disks were estimated by Keller [13]. A method of collocation were applied by Kołodziej & Strek [14] to various periodic arrays of cylinders. McPhedran et al. [15-17] calculated the effective conductivity of arrays of cylinders having used multipole expansions for the fields in and around each inclusion and generalise an identity of Rayleigh [18] to provide a set of linear equations for the multipole coefficients.

There is considerable interest in the effect of polydispersity on the effective conductivity λ_e (see [1] and papers cited therein) The main question is whether the presence of polydispersity impacts on the effective conductivity. Theoretical results point to the increase of λ_e for perfectly conducting inclusions for dilute composites:

$$\lambda_e(poly) > \lambda_e(mono). \quad (1)$$

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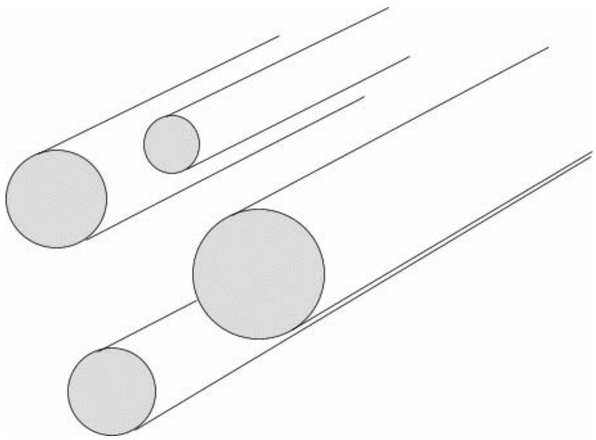

Fig. 1. Cylinders embedded in a host matrix

For densely packed inclusions, the situation is more complicated. Berlyand & Mityushev [1] modelled random inclusions by shaking parameters. They proposed two models: (i) the ``bumping model'', and (ii) ``the well-separated model''. The model (i) describes random geometrical arrays when inclusions are highly packed and not well-separated. The key ingredient of this model is the introduction of two shaking parameters d_l and d_s depending on the sizes of the inclusions for large and small disks, respectively. In the ``the well-separated model'' the shaking parameter is the same for large and small disks. Model (ii) leads to the same conclusion (1). However, the model (i) predicts regimes where either (1) or the opposite inequality

$$\lambda_e(poly) \le \lambda_e(mono) \,. \tag{2}$$

holds, depending on the value of the relative volume fractions of large and small disks.

In this paper, we propose a mathematical model which demonstrate that the both increase and decrease occur and describe the dependence of this effect on the location of inclusions without restrictions made in [1] on the local shaking parameters which allow to locate inclusions in prescribed small cells. Our model corresponds to the uniform non-overlapping distribution of two disks per cell. The proposed model can be extended to arbitrary number N of inclusion per the periodicity cell, however, in the present time we can perform symbolic computations only for the case N = 2. Therefore, our results are obtained under other geometrical conditions, but they confirm the same physical conclusions of [1].

2. Statement of the problem

In the present section, the problem is stated in terms of complex potentials by means of the normalised dimensionless parameters [2-10]. We use the complex variable $z = x + i\,y$, when the section of the composite lies in the plane (x,y) which is perpendicular to the cylinder directions. Consider non-overlapping disks $D_k = \{z \in C : |z - a_k| < r_k$ (k = 1, 2, ..., N) in the rectangular periodicity cell Q_0 of the sizes $\alpha \times \alpha^{-1}$ on the complex plane C.

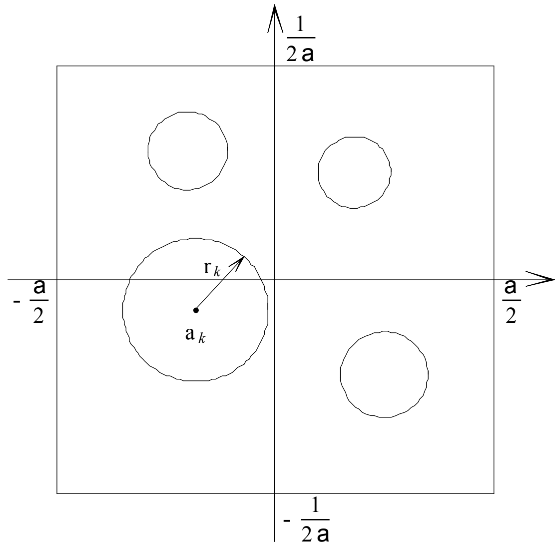

Fig. 2. Periodicity cell Q_0 with circular inclusions

Let D denote the complement of the closures of D_k to Q_0 (see Fig.1). The area of Q_0 is equal to unity. As an example, we will take the square cell Q_0 when α = 1. According to the properties of the effective thermal conductivity tensor, it does not depend on the scale of the representative cell, hence the unit area of Q_0 does not mean any restriction on the geometry of the fibre composite. Therefore, the radii r_k are considered respectively to the lines sizes of the periodicity cell.

Let the domains D and D_k are occupied by materials of the normalised conductivities 1 and λ, respectively. The temperature distribution $T(x,y) = T(z)$ satisfies Laplace's equation $\nabla^2 T = 0$ in the disks D_k and in the matrix D with the conjugation conditions (transmission conditions)

$$T^+ = T^-, \quad \frac{\partial T^+}{\partial n} = \lambda \frac{\partial T^-}{\partial n}, \quad |z - a_k| = r_k, \; k = 1, 2, ..., N, \tag{3}$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative to the circle $|z - a_k| = r_k$ and T^+, T^- denote the limit values of the temperature distribution on the circle from matrix and from inclusions, respectively. Let the external field is applied in the x-direction. The geometry of the fibre composite presented in Fig.1 is doubly periodically continued onto whole plane with the conjugation conditions (3). Therefore, α and $i\alpha^{-1}$ are the fundamental translation vectors expressed in the form of complex numbers. The temperature distribution $T(z)$ is quasi periodic on the complex plane

$$T(z+\alpha) - T(z) = \alpha \,, \qquad T(z + i\alpha^{-1}) - T(z) = 0 \,. \tag{4}$$

The first relation (4) means that the jump of the temperature distribution is equal to α along the *x*-direction per cell. The second relation (4) means that the temperature distribution is periodic in the *y*-direction ($z = x + i\, y$). The heat flux $q = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}\right)$ is doubly periodic on the complex plane

$$q(z+\alpha) - q(z) = 0, \qquad q(z + i\alpha^{-1}) - q(z) = 0. \tag{5}$$

3. Reduction of the problem to functional equations

Introduce the complex potentials $\psi(z)$ and $\psi_k(z)$ analytic in D and D_k, respectively, continuous in their closures in such a way that

$$\frac{\partial T}{\partial x} - i\frac{\partial T}{\partial y} = \begin{cases} \psi(z) - 1 & in \quad D, \\ \frac{2}{\lambda+1}\psi_k(z) \; in \; D_k, & \mathrm{k} = 1, 2, \ldots, \mathrm{n} \end{cases} \tag{6}$$

It is convenient to write the heat flux $q = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}\right)$ as the complex value $q = \frac{\partial T}{\partial x} + i\frac{\partial T}{\partial y}$. Then in the force of (6) it can be written in terms of the complex potentials as follows

$$q(z) = \begin{cases} \overline{\psi(z)} - 1 & in \quad D, \\ \frac{2\lambda}{\lambda+1}\overline{\psi_k(z)} \; in \; D_k, & \mathrm{k} = 1, 2, \ldots, \mathrm{N} \end{cases} \tag{7}$$

where the bar stands for the complex conjugation. Here, the external flux is defined by the vector (-1,0). It is expressed by the complex number $-1 + i0$ (compare to Rayleigh's approach of the same facts in discrete form [15-19]).

It follows from (5) that $\psi(z)$ is doubly periodic on the complex plane

$$\psi(z+\alpha) - \psi(z) = 0, \qquad \psi(z + i\alpha^{-1}) - \psi(z) = 0. \tag{8}$$

Let r denote the maximal radius of r_k. Introduce the dimensionless parameters ξ_k from the segment [0,1] in such a way that $r_k = \xi_k^2 r$. We apply expansion of the complex potentials $\psi_m(z)$ in the parameter r

$$\psi_m(z) = \sum_{l=0}^{\infty} \psi_m^{(l)}(z) r^{2l}, \tag{9}$$

where each function $\psi_m^{(l)}(z)$ is presented in the form of its Taylor series

$$\psi_m^{(l)}(z) = \sum_{n=0}^{\infty} \psi_{nm}^{(l)}(z - a_k)^n. \tag{10}$$

Then the problem (3) is reduced to the system of functional equations [1-10,19] with respect to the functions $\psi_m^{(l)}(z)$ analytic in the disks D_m and continuous in their closures

$$\sum_{l=0}^{\infty} \psi_m^{(l)}(z) r^{2l} = \rho \sum_{k=1}^{N} \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} \overline{\psi_{kn}^{(s)}} \xi_k^n E_{n+2}(z - a_k) r^{2(k+n)} + 1, \quad |z - a_m| < r_m, \quad m = 1, 2, \ldots, N, \tag{11}$$

where ρ is contrast Bergman's parameter introduced in the following way [11,12]

$$\rho = \frac{\lambda - 1}{\lambda + 1}. \tag{12}$$

$E_n(z)$ is Eisenstein's function of order *n* [7,19-22]. The functions $E_n(z)$ ($n \geq 2$) are analytic in the periodicity cell Q_0 except zero, where they have poles of the order *n* with the residuum 1. The functions $E_n(z)$ ($n \geq 2$) are doubly periodic on the complex plane in accordance with the periodicity cell Q_0, i.e., their values coincide at the opposite sides of Q_0. It is assumed that in the sum on *k* from (11) in the term $k = m$ the Eisenstein's function $E_{n+2}(z - a_k)$ is replaced by $E_{n+2}(z - a_m) - \frac{1}{(z-a_m)^{n+2}}$. The effective formulas to compute the values of $E_n(z)$ are given in [7].

Formula (11) have the following physical interpretation. It means that the flux at the *m*-th inclusions is cancelled by the fluxes at all rest inclusions and by the external flux.

The complex potential $\psi(z)$ is expressed through $\psi_k(z)$ by simple formulas given in [1-10]. We do not need these formulas, because the components of the effective conductivity tensor

$$\Lambda = \begin{pmatrix} \lambda_x & \lambda_{xy} \\ \lambda_{xy} & \lambda_y \end{pmatrix} \tag{13}$$

can be exactly written by the values of the flux at the centres of the disks D_k as follows

$$\lambda_x - i\lambda_{xy} = 1 + 2\rho\pi \sum_{k=1}^{N} r_k^2 \psi(a_k). \tag{14}$$

The component λ_y of the symmetric tensor Λ can be determined from other boundary value problem when the external heat flux is applied in a direction not parallel to the *x*-axis, for instance along the *y*-axis. The simplest way to determine λ_y is to change the geometry by rotation about 90^0. Then α is replaced by α^{-1}, the centres a_k by ia_k and the formula (14) becomes a formula for $\lambda_y + i\lambda_{xy}$. Below we use these manipulations to compute the effective conductivity tensor up to $O(\nu^4)$.

It is convenient to use formula (14) in the form

$$\lambda_x - i\lambda_{xy} = 1 + \frac{2\rho\nu}{N\Omega} \sum_{k=1}^{N} \xi_k \psi(a_k), \tag{15}$$

where

$$\nu = N\pi \sum_{k=1}^{N} r_k^2 \quad (16)$$

denote the total concentration of the inclusions in the composite,

$$\Omega = \frac{1}{N}\sum_{k=1}^{N} \xi_k \ . \quad (17)$$

In the case of perfectly conducting inclusions one can assume that $\lambda = +\infty$, hence in the force of (12) $\rho = 1$ and formula (15) becomes

$$\lambda_x - i\lambda_{xy} = 1 + \frac{2\nu}{N\Omega}\sum_{k=1}^{N} \xi_k \psi(a_k) \ . \quad (18)$$

4. Two inclusions per cell

In order to compute the effective conductivity tensor we use formulas (15) or (18), where the complex potentials $\psi_m(z)$ can be found by formulas (9)-(11). The crucial step is to calculate $\psi_m(z)$, i.e., to solve the functional equation (11). A method of successive approximations for (11) had been proposed in [1-10] in the following form. Selecting the coefficients on the same powers of r^2 we arrive at the recurrence formulas for $\psi_m^{(l)}(z)$: the zero approximation is given as $\psi_m^{(0)}(z) = 1$ for all m = 1, 2, ..., N, the next functions $\psi_m^{(l)}(z)$ are exactly calculated through the precedent approximations $\psi_m^{(q)}(z)$, q = 1, 2, ..., l – 1. It is worth noting that all computations are symbolic, hence the results involve all parameters such as ρ, r_k, a_k in symbolic form. Hence, we obtain an approximate analytical formula for the heat flux at the inclusions in the form of complex potentials for arbitrary distributions of the geometrical parameters r_k, a_k. Further, the values $\psi_m(a_m) = \sum_{l=0}^{\infty} \psi_m^{(l)}(a_m) r^{2l}$ have to be calculated and substituted into (15) or (18). Ultimately, we obtain the components of the effective conductivity tensor in analytic form. The convergence of the method of successive approximations for (11) had been proven in [1-10] for any contrast parameter and for any non-overlapping location of the disks in the periodicity cell.

In the present section, we apply this method for two perfectly conducting inclusions when N = 2 and ρ = 1. Then the above method yields formula

$$\lambda_x - i\lambda_{xy} = 1 + 2\nu(1 + C_1\nu + C_2\nu^2 + ...) \ , \quad (19)$$

where the coefficients C_1 and C_2 are determined by the following formulas

$$C_1 = \frac{1}{\pi(\xi_1+\xi_2)} \sum_{k_0,k_1} \xi_{k_0}\xi_{k_1} E_2(a_{k_0} - a_{k_1}) \ , \quad (20)$$

$$C_2 = \frac{1}{\pi^2(\xi_1+\xi_2)^2} \sum_{k_0,k_1,k_2} \xi_{k_0}\xi_{k_1}\xi_{k_2} \overline{E_2(a_{k_0} - a_{k_1})} E_2(a_{k_1} - a_{k_2}) \ . \quad (21)$$

In the sums from (20)-(21), the integers k_0, k_1, k_2 take the values 1 and 2. Using this fact we simplify (20)-(21). Let the first disk has the maximal radius r and the second disk has the radius $r\sqrt{\eta}$, i.e., $\xi_1 = 1$ and $\xi_2 = \eta$. Let us note that (20)-(21) do not depend on the locations of the centres, but they depend on the difference $a = a_1 - a_2$. Then (20)-(21) can be written in the form

$$C_1 = \frac{1}{\pi(1+\eta)^2}[(1+\eta^2)S_2 + 2\eta E_2(a)] \ , \quad (22)$$

$$C_2 = \frac{1}{\pi^2(1+\eta)^2}[(1-\eta+\eta^2)S_2^2 \operatorname{Re} E_2(a) + \eta |E_2(a)|^2] \ , \quad (23)$$

where Re stands for the real part of the complex value. The parameter S_2 is defined as the zero coefficient in Laurents's expansion of the function $E_2(z)$ near the point $z = 0$. The following constructive formula is deduced in [19]

$$S_2 = \frac{\pi^2}{\alpha^2}\left(\tfrac{1}{3} - 2\sum_{m=1}^{\infty} \sinh^{-2}(\pi m \alpha^{-2})\right).$$

It was also proven in [18,19] that $S_2 = \pi$ for the square array when $\alpha = 1$. One can calculate the next coefficients C_j. However, as we will see below it is enough to have two coefficients in the expansion (19) to capture the effect of polydispercity.

Substituting (22)-(23) into (19) we arrive at the relation

$$\lambda_x - i\lambda_{xy} = 1 + 2\nu\{1 + \frac{\nu}{\pi(1+\eta)^2}[(1+\eta^2)S_2 + 2\eta E_2(a)] + \quad (24)$$

$$\frac{\nu^2}{\pi^2(1+\eta)^2}[(1-\eta+\eta^2)S_2^2 \operatorname{Re} E_2(a) + \eta |E_2(a)|^2] \} + O(\nu^4).$$

This is the main formula in our study. Applying the arguments described at the end of Sec.3 and the results of [19] we obtain from (24)

$$\lambda_y + i\lambda_{xy} = 1 + 2\nu\{1 + \frac{\nu}{\pi(1+\eta)^2}[(1+\eta^2)(2\pi - S_2) + 2\eta E_2(ia)] + \quad (24')$$

$$\frac{\nu^2}{\pi^2(1+\eta)^2}[(1-\eta+\eta^2)(2\pi - S_2)^2 \operatorname{Re} E_2(ia) + \eta |E_2(ia)|^2] \} + O(\nu^4).$$

Consider the monodisperse fibre composite when $\eta = 1$. Then (24) becomes

$$\lambda_x - i\lambda_{xy} = 1 + 2\nu\{1 + \frac{\nu}{2\pi}[S_2 + E_2(a)] + \quad (25)$$

$$\frac{\nu^2}{4\pi^2}[(S_2^2 + 2S_2 \operatorname{Re} E_2(a) + |E_2(a)|^2] \} + O(\nu^4).$$

For the square array of cylinders for monodisperse fibre composite we have

$$\lambda_x - i\lambda_{xy} = 1 + \nu\{1 + \nu\left[1 + \frac{E_2(a)}{\pi}\right] + \frac{\nu^2}{2}\left[1 + 2\pi \operatorname{Re} E_2(a) + |E_2(a)|^2\right] \} + O(\nu^4). \quad (26)$$

Let us note that (24) implies that

$$\lambda_{xy} = \frac{4\eta\nu^2}{\pi(1+\eta)^2}\operatorname{Im} E_2(a) + O(\nu^4). \qquad (26)$$

Therefore, it follows from the properties of Eisenstein's function $E_2(z)$ that $\lambda_{xy} = O(\nu^4)$ if and only if a is a real or pure imaginary number. This can be treated as the plane isotropy of the discussed fibre composite up to $O(\nu^4)$. In particularly, the principal axes of the effective conductivity tensor (13) coincide with the axes *OX* and *OY* if and only if the segment connecting the centres of the inclusions is parallel to the axes *OX* and *OY*.

Below we present the results of computations in according to the above formulas (20)-(26) in the graphical form.

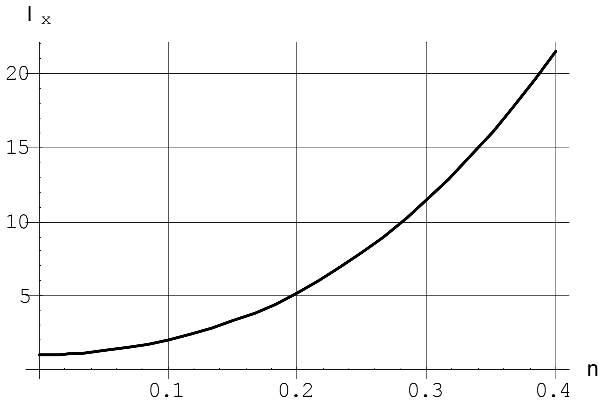

Fig. 3. Dependence of the effective conductivity on the concentration ν for two equal inclusions with fixed $a = 0.5$

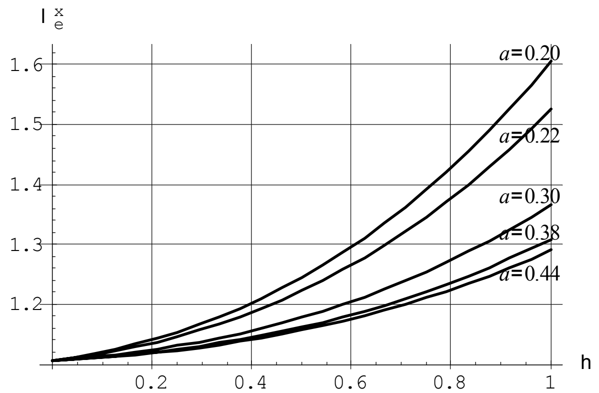

Fig. 4. Dependence of the effective coefficient on η for different a

All computations are made for the square array of cylinders. One can see in Fig.3 that the concentration is the most essential factor of the increasing of the conductivity.

The polydispersity factor η impacts on the effective conductivity as shown at Figs. 4-7. It follows from Fig.4 that the effective conductivity increases when η increases and when $|a|$ decreases. The dependence on complex $a = x + iy$ is presented in Figs. 5-6. The symmetry of the effective conductivity follows from Fig. 7. This figure also demonstrates that the minimal value of λ_x is attached for the equal inclusions (monodispersity with $\eta = 1$).

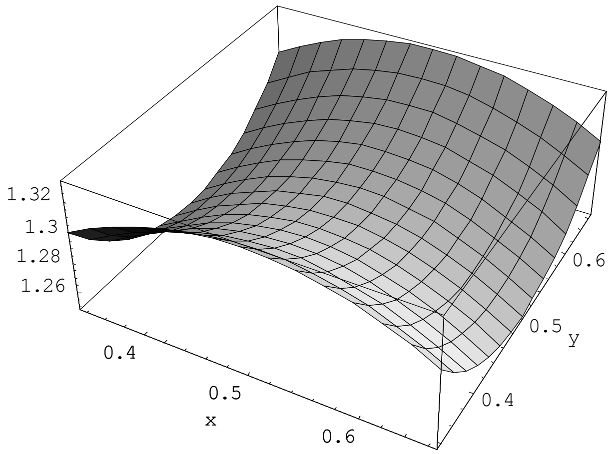

Fig. 5. Dependence of the effective coefficient on $a = x + iy$ for $\nu = 0.1$ and $\eta = 0.415$

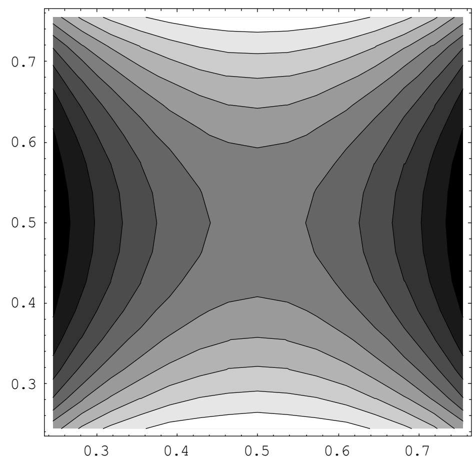

Fig. 6. Dependence of the effective coefficient on $a = x + iy$ for $\nu = 0.1$ and $\eta = 0.415$ on level plot. Dark intensity corresponds to the height of the surface from black to white

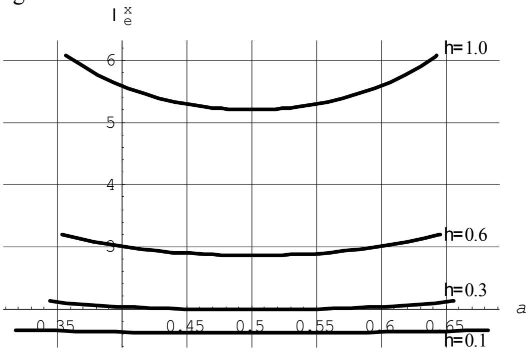

Fig. 7. Dependence of the effective coefficient on a for fixed $\nu = 0.1$ and different η

Consider two inclusions in the square array when one of the inclusion is fixed at the centre of the cell and the second inclusion changes its position i such a way that the distance between the centres of the inclusions stays constant (see Fig. 8)

Let us fix the concentration and the distance. Changing the angle θ yields the change of the effective conductivity. Typical examples of this dependence are presented in Figs. 9-13.

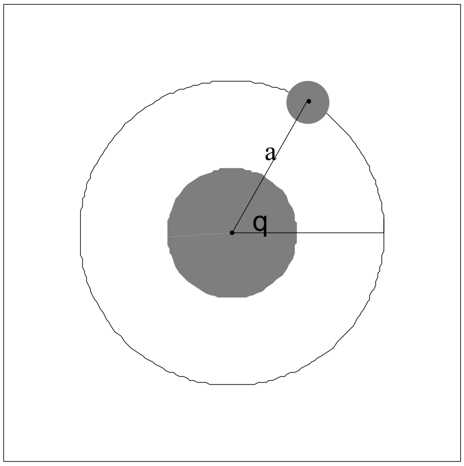

Fig. 8. Two inclusions with fixed distances between centres

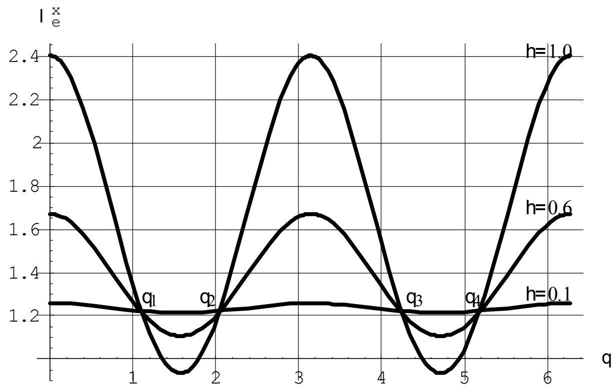

Fig. 9. Dependence of the effective coefficient on θ for different η; concentration $\nu = 0.1$, $|a| = 0.3$

On can see from Fig.9 that there are exist such four angles θ for which λ_x takes the same value independently on η. The behavior of λ_x is different at different segments of θ bounded the four exceptional values. It follows from Figs. 9-10 that for small concentrations the minimal and the maximal values of λ_x are not changed. However, the features of the plots are different. It is related to the effect of interactions of the inclusions.

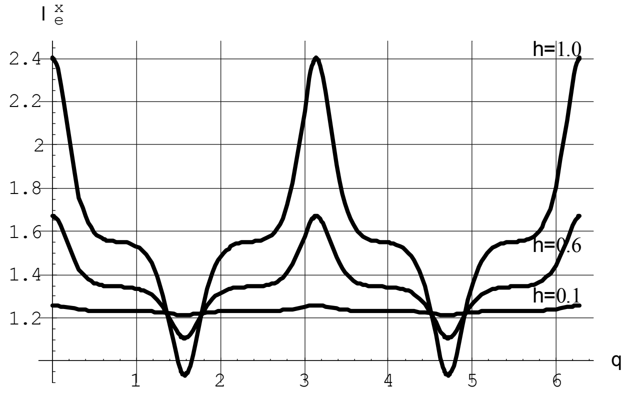

Fig. 10. Dependence of the effective coefficient on θ for different η; concentration $\nu = 0.1$, $|a| = 0.7$

Similar behavior of λ_x is presented in Fig.11-12 except the effect of the "horizontal zones" which disappear for the considered higher parameters $|a|$.

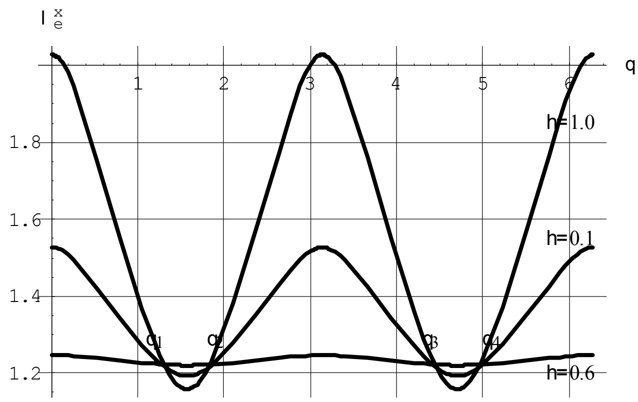

Fig. 11. Dependence of the effective coefficient on θ for different η; concentration $\nu = 0.1$, $|a| = 0.45$

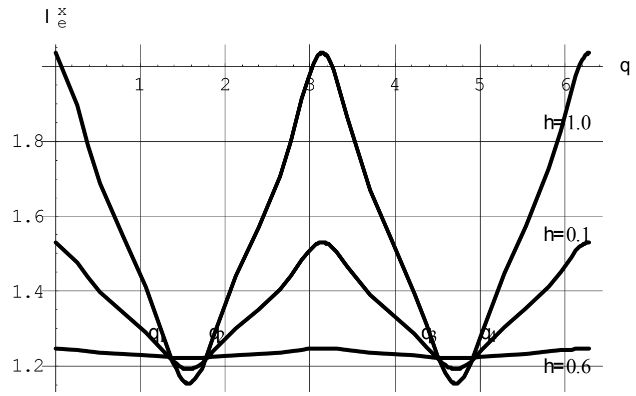

Fig. 12. Dependence of the effective coefficient on θ for different η; concentration $\nu = 0.1$, $|a| = 0.56$

One can observe the dependence of λ_x on θ for different concentrations $\nu = 0.1$ and $\nu = 0.3$ in Fig. 13. The critical values $\theta_1 = 1.3$, $\theta_2 = 1.84$, $\theta_3 = 4.44$, $\theta_4 = 4.98$ (in radians) do not depend on the concentration and the polydispersity parameter (*a* is fixed). In practice, this implies the existence of such directions in which λ_x depends only on the total concentration, not on the sizes of the inclusions.

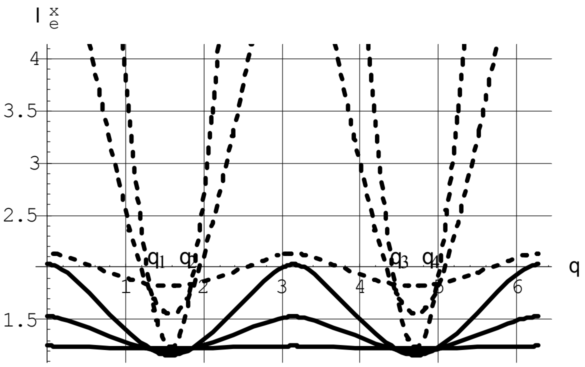

Fig. 13. Dependence of the effective coefficient on θ for different η and for $\nu = 0.1$ (solid lines), $\nu = 0.3$ (dashing lines); $|a| = 0.45$

In order to investigate the quantitative effect of polydispersity in details introduce the invariants of the tensor Λ

$$I_1 = \lambda_x + \lambda_y, \quad I_2 = \det \Lambda .$$

Using (24) and (24') for the square array we obtain

$$I_1(\eta) = 1 + \nu\{2 + \frac{2\nu(1+\eta^2)}{(1+\eta)^2} + \frac{2\nu\eta}{\pi(1+\eta)^2}\,\mathrm{Re}[E_2(a) + E_2(ia)] + \qquad (27)$$

$$\frac{\nu^2(1-\eta+\eta^2)}{(1+\eta)^2} + \frac{\nu^2\eta}{\pi^2(1+\eta)^2}\left(|E_2(a)|^2 + |E_2(ia)|^2\right) + O(\nu^4) .$$

Let the polydispersity parameter η changes on the segment (0,1]. In order to compare $I_1(\eta)$ with $I_1(0)$ up to $O(\nu^3)$ consider the increment of the invariant

$$\Delta I_1 = I_1(\eta) - I_1(1) . \qquad (28)$$

The value ΔI_1 as a function of ν is analytic near zero. Hence it can be expanded into a power series on the concentration ν with coefficients ΔC_j depending on η

$$\Delta I_1 = \Delta C_0 + \Delta C_1 \nu + \Delta C_2 \nu^2 + \Delta C_3 \nu^3 + \dots .$$

We have $\Delta C_0 = \Delta C_1 = 0$, since the coefficients of $I_1(\eta)$ up to ν^2 do not contain η. The coefficient ΔC_2 has the form

$$\Delta C_2 = \left(\frac{1-\eta}{1+\eta}\right)^2 \left(1 - \frac{1}{2\pi}[E_2(a) + E_2(ia)]\right) .$$

Using the properties of Eisenstein's functions [7,19-22] we obtain that

$$E_2(a) + E_2(ia) = 0$$

for the square array. Hence, $\Delta C_2 = 0$. Thus, we have proved that the power series of ΔI_1 on ν begins from the term ν^3, more precisely, we get up to $O(\nu^4)$

$$\Delta I_1 = \frac{\nu^3}{2}\left(\frac{1-\eta}{1+\eta}\right)^2 f(a), \qquad (29)$$

where the real function $f(a)$ of the complex argument $a = x + iy$ is given by formula

$$f(a) = 1 - \frac{1}{2\pi^2}\left(|E_2(a)|^2 + |E_2(ia)|^2\right) . \qquad (30)$$

Therefore, the sign of ΔI_1 coincided to the sign of the function $f(a)$ presented in Fig.14 in a set of the lines $y = y(x)$ corresponding to different values of *a*. For instance, $y = x$ corresponds to the values $f(x+ix)$ calculated by (30) with $a = x + ix$. The curves corresponding to $y = x$, $y = 0.7x$, $y = 0.5x$ pass the line $f(x+iy) = 0$. Hence, ΔI_1 changes the sign and inequality (2) is replaced by (1). Two curves lying below $f(a) = 0$ correspond to such locations of the inclusions that only inequality (2) takes place.

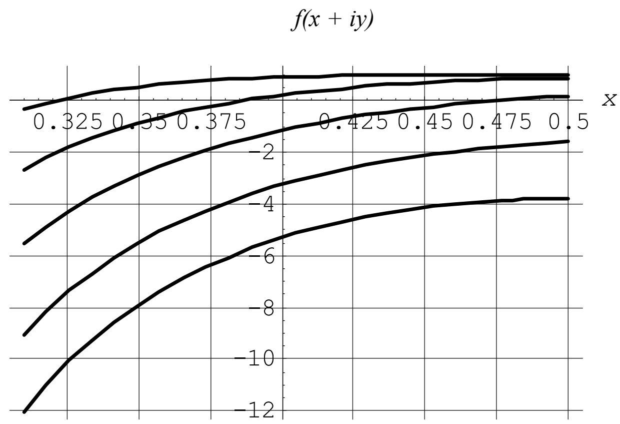

Fig. 14. Dependence of $f(x + iy)$ on $a = x + iy$. The curves from top to bottom correspond to the lines $y = x$, $y = 0.7x$, $y = 0.5x$, $y = 0.3x$, $y = 0$ in the square cell

5. Conclusions

In this paper we discuss the question of polydispercity for fibre composites made from unidirectional circular cylinders arbitrary distributed in a uniform host applying the theory [1-10]. For definiteness, the case of perfectly conducting inclusion is considered. We have established that the both inequality (1) and (2) can occur.

It is important to note that inequality (1) were suggested by many previous scientists, i.e., it was thought that in order to obtain a composite with higher effective conductivity it is better to use inclusions of various sizes. However, beginning from the paper [1] it was checked that the suggestion (1) is valid for dilute composites. It is clear from our formula (29) where the difference between the first invariants of the effective conductivity tensor $\Delta I_1 = I_1(poly) - I_1(mono)$ is proportional to ν^3, where ν is the total concentration of inclusions. The coefficient of proportionality is written in closed form (29)-(30). Direct computations show in Fig.14 that the value ΔI_1 can take different signs. This is the simplest demonstration that the both inequality (1) and (2) can occur.

The detailed investigation of the polydispersity effect is based on the approximate analytical formulas (24)-(26) in which the components of the effective conductivity tensor are explicitly written up to $O(\nu^4)$. It is important that formulas (24)-(26) contain the geometrical parameters of the problem (locations of the centres of inclusions, their radii) in symbolic form. This allows us systematically study the dependence of the components of the effective conductivity tensor on geometry. The results are presented in Figs. 3-14. They demonstrate in details many interesting properties of the complicated structures under discussion.

Experimental results are presented in [22,23] and works cited therein.

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